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# NUMERICAL SOLUTION OF SOME PROBLEMS OF BOUNDARY-LAYER THEORY

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**Аннотация**—В статье излагаются результаты численного решения ряда задач теории пограничного слоя несжимаемой жидкости и сжимаемого газа. С помощью статического электроинтегратора получены решения задач об обтекании пластины однородным потоком вязкой несжимаемой жидкости с постоянной и переменной вязкостью, об истечении плоской струи несжимаемой жидкости из насадка конечного размера, о распространении полуограниченной струи несжимаемой жидкости вдоль цилиндра и пограничном слое в сжимаемом газе на непрерывной движущейся плоской поверхности. Результаты решения иллюстрируются графиками.

## NOMENCLATURE

- $U, \vartheta$, longitudinal and transverse components of velocity;
- $x, y$, Cartesian co-ordinates;
- $\nu$, kinematic viscosity;
- $\rho$, density;
- $V$, electrical potential;
- $T$, temperature;
- $\theta$, dimensionless temperature;
- $k$, thermal diffusivity;
- $l$, half-width of channel;
- $R, R_0$, resistances;
- $h$, enthalpy;
- $\xi, \eta$, Dorodnitsyn variables;
- $M_w$, Mach number.

Subscripts
- $w$, wall;
- $\infty$, far away from the wall.

## 1. INTRODUCTION

THE ANALYTICAL solution of the boundary-layer equations can only be obtained in some cases. This is explained by insuperable (at least nowadays) mathematical difficulties due to integration of non-linear partial differential equations.

Various numerical solutions are therefore of great interest. Among them a finite-difference method has been widely used, especially recently, which makes it possible to obtain a solution as accurate as required by comparatively simple calculations. However, when solving the sets of equations with various non-linearities, complex boundary conditions, certain singularities, etc., it is not always possible to find a logically simple algorithm of calculation, which makes programming rather difficult.

Therefore in addition to the digital units (and sometimes together with them) it is advisable to use analogue computers whose operation is based on a mathematical modelling of the initial equations. In this paper the results are discussed of the solution of some problems in boundary-layer theory with the aid of statical electro-integrators designed at the Kazakh State University [1–3].*

Using the statical electro-integrators, the solution is performed in a grid region step by step, which allows one to control the calculations, to stop them, or introduce alterations, etc.

* Statistical electro-integrator, type "СЭИ-I" is at present produced by Kazakh Council of National Economy.

This makes it possible to pass easily from one scheme of computation to another, to choose optimum variants, to watch the stability of the computations, etc.

The principle of the integrator operation is based on mathematical modelling with the help of an electric system of differential equations written in terms of finite differences. The solution is performed in discrete co-ordinates and the grid mesh chosen with an eye on the solution stability. The integrator (Fig. 1) consists of a number of discrete functional potentiometers of great resolving power. The change of equation coefficients is programmed by the potentiometers before the computation is started. The terminals of the potentiometers are fixed on a common panel of commutation. The calculation is performed by moving one solving element over the points of spatial region represented in the integrator by potentials. A solving element consists of several resistors (sometimes capacitors) connected in a predefined circuit. The solution is sought on the same panel of commutation by the zero-method.

Some solutions of problems obtained on statical electro-integrators are given below.

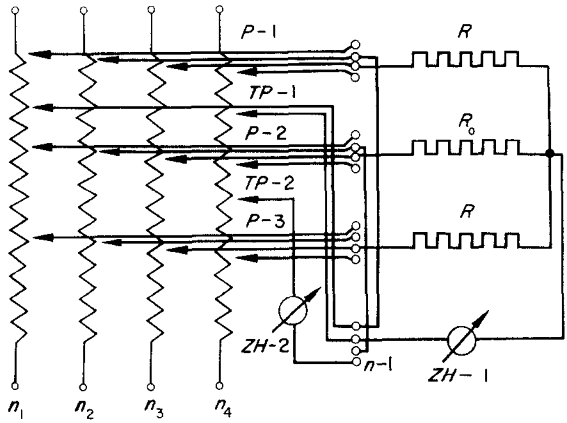


FIG. 1. Electric circuit of statical electro-integrator. $n_1$, $n_2$, $n_3$ are functional potentiometers; R, $R_0$, are resistors of solving element; ZH, zero galvanometer; TP, test plug; P, plug.

## 2. VISCOUS INCOMPRESSIBLE FLOW AROUND A PLATE

The problem is written mathematically in the form of the following set of differential equations and boundary conditions

$$U\frac{\partial U}{\partial x} + \vartheta\frac{\partial U}{\partial y} = \nu\frac{\partial^2 U}{\partial y^2}, \qquad \frac{\partial U}{\partial x} + \frac{\partial \vartheta}{\partial y} = 0; \tag{1}$$

$$U = \vartheta = 0 \quad \text{at} \quad y = 0, \qquad U = U_\infty, \quad \frac{\partial U}{\partial y} = 0 \quad \text{at} \quad y = \infty. \tag{2}$$

The corresponding system of finite difference dimensionless equations is of the form:

$$\frac{U_{n,k+1} - U_{n,k}}{\Delta x} = \frac{\nu}{\Delta y^2}\frac{1}{a}\sum_{i=1}^{a}\frac{1}{U_i}(U_{n-1,k} - 2U_{n,k} + U_{n+1,k}) - \frac{1}{b}\sum_{i=1}^{b}\frac{\vartheta_i}{U_i}\frac{1}{2\Delta y}(U_{n-1,k} - U_{n+1,k}),$$

$$\vartheta_{n,k+1} = \vartheta_{n,k} + U_{n,k} - U_{n,k+1}, \tag{3}$$

where

$$a = \frac{U_{n,k+1} - U_{n,k}}{N}, \qquad b = \frac{U_{n-1,k} - U_{n+1,k}}{N};$$

$N = 0{\cdot}001$ is the smallest value of the function when the change of the coefficients is taken into account. Equations (3) are written allowing for the fact that the region of the variation of the function is divided into two grids. The first is a fixed one with a mesh of $\Delta x \neq \Delta y$ selected from the stability conditions of the solution, the second (mobile) has a mesh of $\Delta x = \Delta y$.

The equality

$$V_{n,k+1} - V_{n,k} = \frac{1}{2 + R/R_0}\frac{1}{a}\sum_{i=1}^{a}\frac{1}{V_i}(V_{n-1,k} - 2V_{n,k} + V_{n+1,k}) - A\frac{1}{b}\sum_{i=1}^{b}V_i'(V_{n-1,k} - V_{n+1,k}) \quad (4)$$

is valid for the integrator scheme (Fig. 1).

The comparison of equations (3) and (4) yields the conditions of modelling

$$\frac{\nu\Delta x}{\Delta y^2} = \frac{1}{2 + R/R_0}; \qquad \frac{1}{a}\sum_{i=1}^{a}\frac{1}{U_i} = \frac{1}{a}\sum_{i=1}^{a}\frac{1}{V_i}; \qquad \frac{1}{b}\sum_{i=1}^{b}\frac{\vartheta_i}{U_i} = \frac{1}{b}\sum_{i=1}^{b}V_i'; \qquad \frac{1}{2\Delta y} = A. \quad (5)$$

Conditions (5) are satisfied on the integrator (before the solution starts) by setting the necessary ratio between the resistors of a solving element $R/R_0$, by programming on the potentiometers the

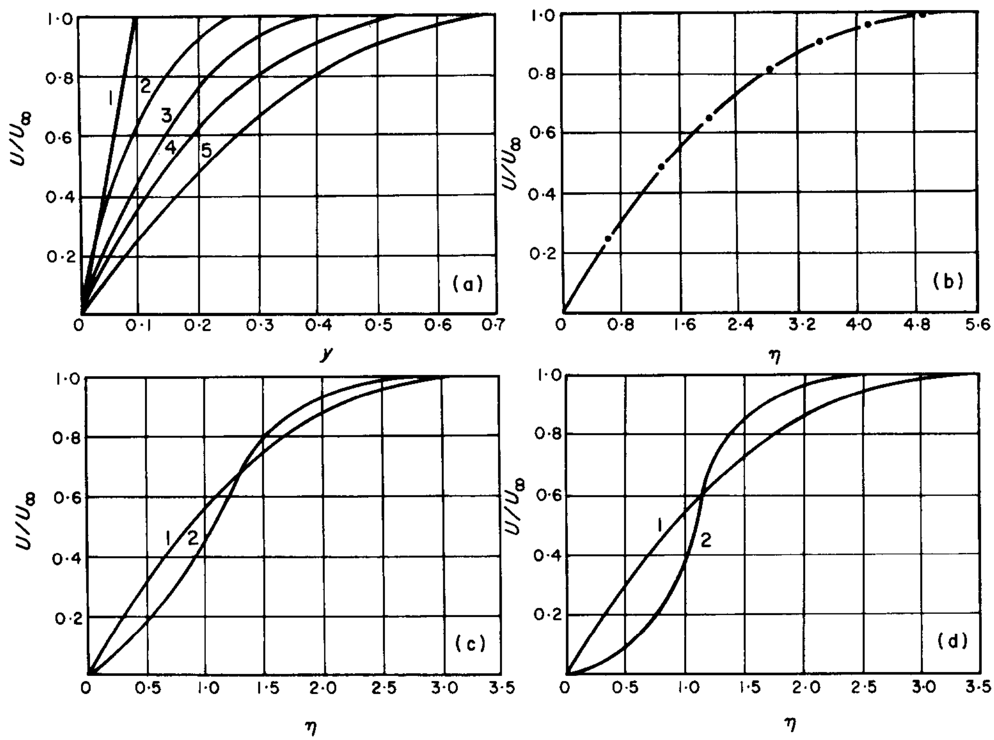


FIG. 2.

(a) Development of the velocity profile when incompressible viscous fluid is flowing around a plate. Curve 1 is the initial profile. Profile 5 and the following ones (not shown in the Figure) are similar profiles.

(b) Comparison of a similar profile obtained by the integrator (points) with that of Blasius (solid curve).

(c) Similar velocity profiles in boundary layer for water flow around a plate. 1.—$T_w = 80°C$, $T_\infty = 20°C$; 2.—$T_w = 20°C$, $T_\infty = 80°C$.

(d) Similar velocity profiles in boundary layer for lubricating oil flow around a plate. 1.—$T_w = 90°C$, $T_\infty = 50°C$; 2.—$T_w = 50°C$, $T_\infty = 90°C$.

functional dependences $F_1 = f(U)$ and $F_2 = f(\vartheta/U)$ and by an appropriate selection of the supply current for the potentiometers.

The results of the solution are presented in Figs. 2(a) and 2(b). Figure 2(a) shows the development of the velocity profile up to the onset of similarity. In Fig. 2(b) the curve corresponds to the similar velocity profile obtained by Blasius [4], the points correspond to the solution found on the integrator.

### 3. UNIFORM LIQUID FLOW OF VARIABLE VISCOSITY AROUND A PLATE

Assuming that all the properties except viscosity* are constant, the initial system of equations and boundary conditions can be written in the form

$$U\frac{\partial U}{\partial x} + \vartheta\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left(\vartheta\frac{\partial U}{\partial y}\right), \qquad \frac{\partial U}{\partial x} + \frac{\partial \vartheta}{\partial y} = 0 \tag{6}$$

$$U = \vartheta = 0 \quad \text{at} \quad y = 0, \qquad U = U_\infty, \quad \frac{\partial U}{\partial y} = 0 \quad \text{at} \quad y = \infty, \qquad U\frac{\partial T}{\partial x} + \vartheta\frac{\partial T}{\partial y} = k\frac{\partial^2 T}{\partial y^2}, \tag{7}$$

$$T = T_w \quad \text{at} \quad y = 0, \qquad T = T_\infty \quad \text{at} \quad y = \infty.$$

Equations (6) and (7) in an explicit, finite difference form and solved for the unknown functions, will be written:

$$U_{n,k+1} = \frac{\Delta x}{\Delta y^2}\frac{1}{a}\sum_{i=1}^{a}\frac{1}{U_i}\left[\frac{1}{b}\sum_{i=1}^{b} v_i(U_{n+1,k} - U_{n,k}) - \frac{1}{c}\sum_{i=1}^{c} v_i(U_{n,k} - U_{n-1,k})\right] + U_{n,k} - \frac{\Delta x}{\Delta y}\frac{1}{d}\sum_{i=1}^{d}\frac{\vartheta_i}{U_i}(U_{n+1,k} - U_{n,k}), \tag{8}$$

$$\vartheta_{n,k+1} = \vartheta_{n,k} + U_{n,k} - U_{n,k+1},$$

$$\theta_{n,k+1} = \frac{\Delta x}{\Delta y^2 Pr_\infty}\frac{1}{l}\sum_{i=l}^{l}\frac{1}{U_i}(\theta_{n+1,k} - 2\theta_{n,k} + \theta_{n-1,k}) + \theta_{n,k} - \frac{\Delta x}{\Delta y}\frac{1}{f}\sum_{i=1}^{f}\frac{\vartheta_i}{U_i}(\theta_{n+1,k} - \theta_{n,k}), \tag{9}$$

where

$$\theta = \frac{T - T_w}{T_\infty - T_w}, \qquad Pr_\infty = \frac{v_\infty}{k} \quad \text{(Prandtl number)}.$$

One can write for the integrators:

$$V_{n,k+1} = \frac{1}{2 + R/R_0}\frac{1}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{V_i}\left[\frac{1}{\beta}\sum_{i=1}^{\beta} V_i(V_{n+1,k} - V_{n,k}) - \frac{1}{\gamma}\sum_{i=1}^{\gamma} V_i(V_{n,k} - V_{n-1,k})\right. + V_{n,k} - A(V_{n+1,k} - V_{n,k}), \tag{10}$$

* In these calculations the temperature dependence of viscosity found experimentally [6] is used.

$$V_{n,k-1} = \frac{1}{2 + R/R_0}\frac{1}{\delta}\sum_{i=1}^{\delta}\frac{1}{V_i}(V_{n+1,k} - 2V_{n,k} + V_{n-1,k}) + V_{n,k} - B(V_{n-1,k} - V_{n,k}). \quad (11)$$

We find the conditions of modelling by comparing equations (8) and (10), (9) and (11):

$$\frac{\Delta x}{\Delta y^2} = \frac{1}{2 + R/R_0}, \quad \frac{1}{a}\sum_{i=1}^{a}\frac{1}{U_i} = \frac{1}{\alpha}\sum_{i=1}^{\alpha}\frac{1}{V_i}, \quad \frac{1}{b}\sum_{i=1}^{b} V_i = \frac{1}{\beta}\sum_{i=1}^{\beta} V_i,$$

$$\frac{1}{c}\sum_{i=1}^{c} V_i = \frac{1}{\gamma}\sum_{i=1}^{\gamma} V_i, \quad \frac{\Delta x}{\Delta y}\frac{1}{d}\sum_{i=1}^{d}\frac{\vartheta_i}{U_i} = A,$$

$$\frac{\Delta x}{\Delta y^2\, Pr_\infty} = \frac{1}{2 + R/R_0}, \quad \frac{\Delta x}{\Delta y}\frac{1}{f}\sum_{i=1}^{f}\frac{\vartheta_i}{U_i} = B, \quad \frac{1}{l}\sum_{i=1}^{l}\frac{1}{U_i} = \frac{1}{\delta}\sum_{i=1}^{\delta}\frac{1}{V_i}. \quad (12)$$

Inequalities

$$\frac{1}{U}\frac{\Delta x}{\Delta y^2}\frac{1}{Pr_\infty} \leqslant \tfrac{1}{2}, \quad \frac{\Delta x}{\Delta y^2}\frac{1}{u} \leqslant \tfrac{1}{2} \quad (13)$$

should obviously be observed (both in this and other cases) to satisfy the stability conditions of solution [5].

In Fig. 2 (c) are shown similar velocity profiles in a boundary layer formed when hot (curve I) and cold (curve 2) plates are in a uniform flow. The point of contraflection of the curve 2 is a specific feature of these profiles. Similar curves are given for lubricating oil in Fig. 2(d). In this case the velocity profile deformation is still more vividly expressed on a cold plate. Temperature fields are calculated for the same temperatures of the plate and flow in a boundary layer (Fig. 3a). The comparison of the curves obtained for velocity profiles with those in reference [5] shows satisfactory agreement.

## 4. PLANE INCOMPRESSIBLE JET FLOWING OUT OF A NOZZLE OF FINITE DIMENSION

The equations and boundary conditions which describe the distribution of submerged plane-parallel jet flow out of a flat tube $2l$ wide with a parabolic initial profile are of the form

$$U\frac{\partial U}{\partial x} + \vartheta\frac{\partial U}{\partial y} = \vartheta\frac{\partial^2 U}{\partial y^2}, \quad \frac{\partial U}{\partial x} + \frac{\partial \vartheta}{\partial y} = 0, \quad (14)$$

$$\text{at} \quad x = 0: \begin{cases} y < -l, \quad y > +l, \quad U = 0; \\ -l < y < +l, \quad U = U_{m_0}(1 - y^2/l^2), \end{cases}$$

$$\text{at} \quad x > 0: \frac{\partial U}{\partial y} = 0 \quad \text{at} \quad y = 0$$

$$U = 0, \quad \frac{\partial U}{\partial y} = 0 \quad \text{at} \quad y = \pm\infty.$$

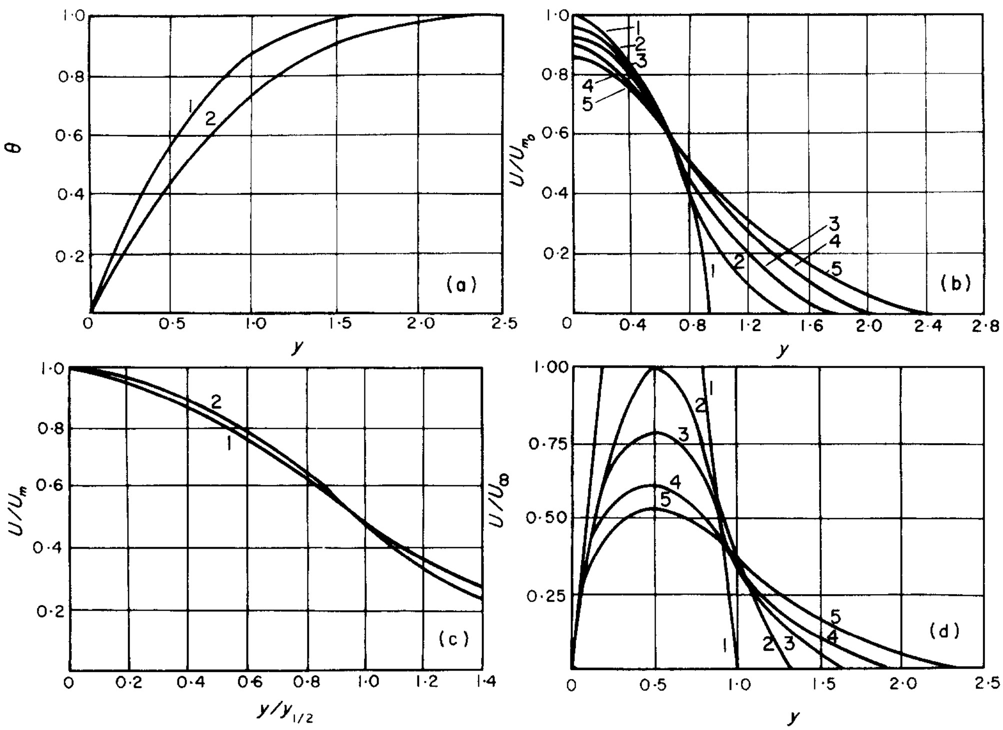


FIG. 3.

(a) Similar temperature profiles in boundary layer for water flow around a plate. 1.—$T_w$ = 80°C, $T_\infty$ = 20°C; 2.—$T_w$ = 20°C, $T_\infty$ = 80°C.

(b) Development of initial (1) parabolic velocity profile in a jet flowing out of a flat tube (curve 5 represents the similar profile).

(c) Comparison of theoretical similar profile $U/U_m = -\tanh^2 \varphi$ (curve 1) with that calculated by the integrator (curve 2).

(d) Velocity distribution at the outflow from an annular channel along a coaxial cylindrical rod. 1 is the initial profile. 5 is profile corresponding to the similar solution.

On using the methods similar to the ones discussed above, one can find the modelling conditions for the integrator and obtain the solution of the system of equations (14).

In Fig. 3(b) the development of the initial parabolic velocity profile is shown. The comparison of the theoretical similar profile $(U/U_m) = 1 - \tanh^2 \varphi$ [4] and that obtained by the integrator is illustrated in Fig. 3(c).

### 5. PROPAGATION OF A SEMI-INFINITE JET ALONG A CYLINDER

The result of the solution of the initial system of equations with appropriate boundary conditions

$$U\frac{\partial U}{\partial x} + \vartheta\frac{\partial U}{\partial y} = \frac{\partial^2 U}{\partial y^2} + \frac{1}{y}\frac{\partial U}{\partial y}, \qquad \frac{\partial U}{\partial x} + \frac{\partial \vartheta}{\partial y} + \frac{\vartheta}{y} = 0;$$
$$U = \vartheta = 0 \quad \text{at} \quad y = 0, \qquad U = \frac{\partial U}{\partial y} = 0 \quad \text{at} \quad y \to \infty \tag{15}$$

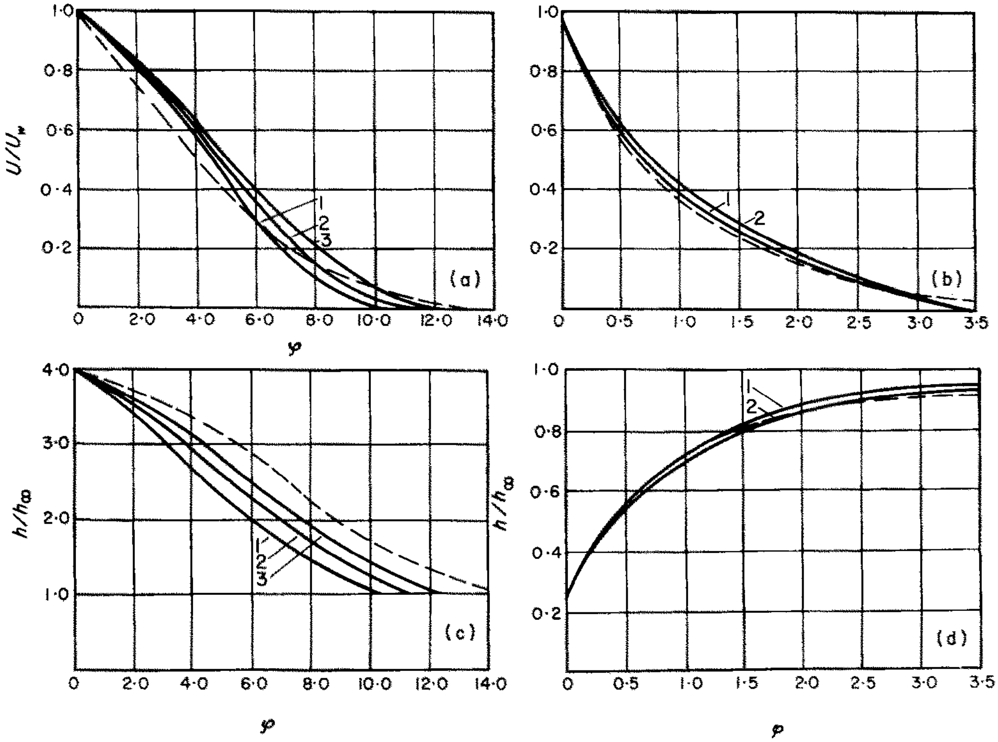


FIG. 4.

(a) Velocity profiles $(U/U_w)$ at $(h_w/h_\infty) = 4$ $(Pr = 0{\cdot}72)$. 1.—$n = 0{\cdot}6$; 2.—$n = 0{\cdot}8$; 3.—$n = 1{\cdot}0$ (the broken line represents the theoretical solution for $n = 1$).

(b) Velocity profiles $(U/U_w)$ at $(h_w/h_\infty) = \frac{1}{4}$ $(Pr = 0{\cdot}72)$. 1.—$n = 0{\cdot}6$; 2.—$n = 0{\cdot}8$ (the broken line represents the theoretical solution for $n = 1$).

(c) Enthalpy profiles at $(h_w/h_\infty) = 4$ $(Pr = 0{\cdot}72)$. 1.—$n = 0{\cdot}6$; 2.—$n = 0{\cdot}8$; 3.—$n = 1{\cdot}0$ (the broken line represents the theoretical solution for $n = 1$).

(d) Enthalpy profiles at $(h_w/h_\infty) = \frac{1}{4}$ $(Pr = 0{\cdot}72)$. 1.—$n = 0{\cdot}6$; 2.—$n = 0{\cdot}8$ (the broken line represents the theoretical solution for $n = 1$).

is given in Fig. 3(d) which illustrates the development of the initial velocity profile before the onset of similarity.

### 6. BOUNDARY LAYER IN COMPRESSIBLE GAS ON A MOVING FLATE PLATE

This problem was studied theoretically for incompressible fluid in papers [6, 7]. In a boundary layer of compressible gas on a continuous flat surface the flow is described by solving the dimensionless equations

$$U\frac{\partial U}{\partial \xi} + \tilde{\vartheta}\frac{\partial U}{\partial \eta} = \frac{\partial}{\partial \eta}\left(h^{n-1}\frac{\partial U}{\partial \eta}\right), \qquad \frac{\partial U}{\partial \xi} + \frac{\partial \tilde{\vartheta}}{\partial \eta} = 0,$$
$$U\frac{\partial h}{\partial \xi} + \tilde{\vartheta}\frac{\partial h}{\partial \eta} = \frac{1}{Pr}\frac{\partial}{\partial \eta}\left(h^{n-1}\frac{\partial h}{\partial \eta}\right) + (k-1)M_0^2 h^{n-1}\left(\frac{\partial U}{\partial \eta}\right)^2, \tag{16}$$

where

$$\tilde{\vartheta} = U\frac{\partial \eta}{\partial x} + \rho\tilde{\vartheta}, \qquad \xi = x, \qquad \eta = \int_0^y \frac{\rho}{\rho_\infty} \mathrm{d}y$$

($\xi$ and $\eta$ are Dorodnitsyn variables) with boundary conditions

$$U = 1, \qquad h = h_0 \qquad \text{at} \quad \eta = 0$$

$$U = 0, \qquad \frac{\partial U}{\partial \eta} = 0, \qquad h = 1, \qquad \frac{\partial h}{\partial \eta} = 0 \qquad \text{at} \quad \eta = \infty.$$

The solution is obtained by the methods already described. In Figs. 4(a) and 4(b) the velocity profiles $U/U_w$ are shown for various values of the parameter $n$ in the physical plane $x, y$. Similar curves are given for enthalpy in Figs. 4(c) and 4(d).

As seen from the above examples, the method proposed for the solution of the boundary-layer theory problems gives qualitatively correct results which are in most cases, sufficiently close to the accurate results obtained by other methods. If necessary the accuracy of the solution can be increased by changing to a smaller grid-mesh. This can readily be done on the integrator and only an increase in the volume of the calculations is involved.

Thus the application of the mathematical modelling method allows the solution of the initial system of partial differential equations for boundary layer to be found, without using any transformations to simplify the system.

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**Abstract**—Results are given of the numerical solution of some problems of boundary-layer theory for incompressible fluid and compressible gas. By a statical electro-integrator the solutions are obtained to the problems of uniform incompressible fluid flow with constant and variable viscosity around a plate; of incompressible plane jet overflow from a nozzle of finite dimension; of propagation of semi-infinite incompressible fluid jet along a cylinder; and of a boundary layer in compressible gas on a continuous moving flat plate. The results are presented in a graphical form.

**Zusammenfassung**—Von einigen Problemen der Grenzschichttheorie für inkompressible Flüssigkeit und kompressibles Gas sind die Ergebnisse einer numerischen Lösung angegeben. Mit Hilfe eines Elektrointegrators wurden die Lösungen für folgende Probleme erhalten: Gleichmässige, inkompressible Strömung um eine Platte bei konstanter und veränderlicher Zähigkeit; inkompressibler, ebener Strahlaustritt aus einer Düse endlicher Dimension; Fortschreiten eines halbunendlichen, inkompressiblen Flüssigkeitsstrahls entlang eines Zylinders und Grenzschichtprobleme an einer kontinuierlich bewegten ebenen Platte in kompressiblem Gas. Die Ergebnisse sind grafisch dargestellt.